In this section, we are going to prove the equivalence between Axiom 3 ("the conjunction of any collection of positive properties is positive") and Proposition 3 ("it is possible that God exists"). First, we have to translate these axiom/proposition which have been stated informally into formal ones. The following is the formal version of Axiom 3.

Axiom 3 $\forall \mathcal{Z}(pos(\mathcal{Z}) \rightarrow \forall X(X \; intersection \; of \; \mathcal{Z} \rightarrow \mathcal{P}(X)))$

Here, $pos(\mathcal{Z})$ is defined by $\forall X(\mathcal{Z}(X) \rightarrow \mathcal{P}(X))$. This informally means that a property $\mathcal{Z}$ is applicable only to positive properties. On the other hand, $X \; intersection \; of \; \mathcal{Z}$ is defined by $\Box \forall x(X(x) \equiv \forall Y(\mathcal{Z}(Y) \rightarrow Y(x)))$.[1] This means, informally again, that a property X is *necessarily* applicable to objects which have properties contained in a "property family" $\mathcal{Z}$. $\mathcal{P}(X)$ means, needless to say, that a property X is positive.

The formal version of the Proposition 3 is as follows:

Proposition 3 $\Diamond \exists x G(x)$

Seeing that $G(x)$ means that "x is God", a further explanation except about the formal definition of $G(x)$ would not be needed. A "godness" property $G(x)$ is, using the lambda notation, defined by $\lambda x.(\forall Y(\mathcal{P}(Y) \rightarrow Y(x)))$ which means an object which has all positive properties.

Before proceeding to a proof that Axiom 3 and Proposition 3 above are actually equivalent, we have to formalize other informal axioms which are needed to prove the equivalence. First, we formalize Axiom 1. This axiom requires a property to be either positive or non-positive (that is, negative), but not both. There are two ways to formalize it.

Axiom 1
A $\forall X(\mathcal{P}(\neg X) \rightarrow \neg \mathcal{P}(X))$
B $\forall X(\neg \mathcal{P}(X) \rightarrow \mathcal{P}(\neg X))$

In order to formalize Axiom 2 which asserts that any properties accompanied by a positive property are also positive, we need a new notation $\mathbf{E}(x)$ and the following abbreviations: $\forall^{\mathbf{E}} x \Phi$ *is the abbreviation for* $\forall x(\mathbf{E}(x) \rightarrow \Phi)$ *and* $\exists^{\mathbf{E}} x \Phi$ *for* $\exists x(\mathbf{E}(x) \wedge \Phi)$.[2] Using this new notation, we can formalize Axiom 2 as follows:

Axiom 2 $\forall X \forall Y((\mathcal{P}(X) \wedge \Box \forall^{\mathbf{E}} x(X(x) \rightarrow Y(x))) \rightarrow \mathcal{P}(Y))$ [3]

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[1] This translation looks somewhat complicated because it deals with a property of properties. An analogy to Axiom of Union in axiomatic set theory — $\forall X \exists Y \forall u(u \in Y \leftrightarrow \exists z(z \in X \wedge u \in z))$ — might be helpful.

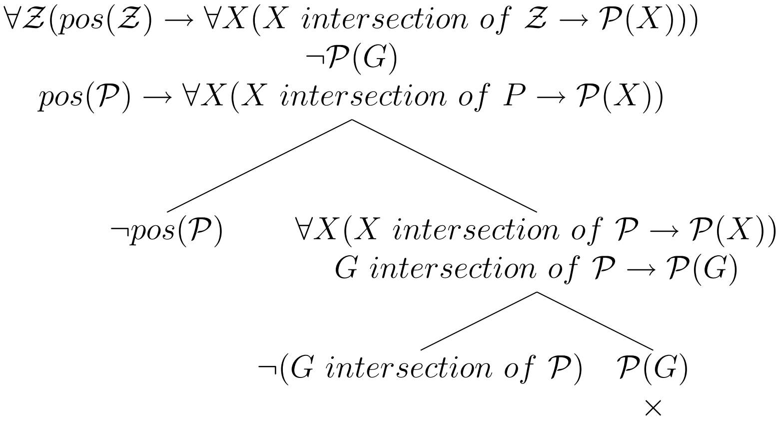
[2] The rationale of this introduction of the new symbol $\mathbf{E}$ is concerning the issue about the two different interpretation of quantifiers in modal logic: *actualist* and *possibilist*. Although the issue is philosophically interesting, we set aside it in this paper.

[3] The necessity symbol $\Box$ is needed here because Gödel thinks that the accompaniment of a property Y to $X$is "the nature of the property" (Gödel, p. 403).

Now we are in a position to prove the equivalence. In proving, instead of directly showing the equivalence between Axiom 3 and Proposition 3, we prove it through the equivalences between the followings:

1. Axiom 3
2. $\mathcal{P}(G)$
3. $\Diamond\exists^{\mathbf{E}}xG(x)$
4. Proposition 3

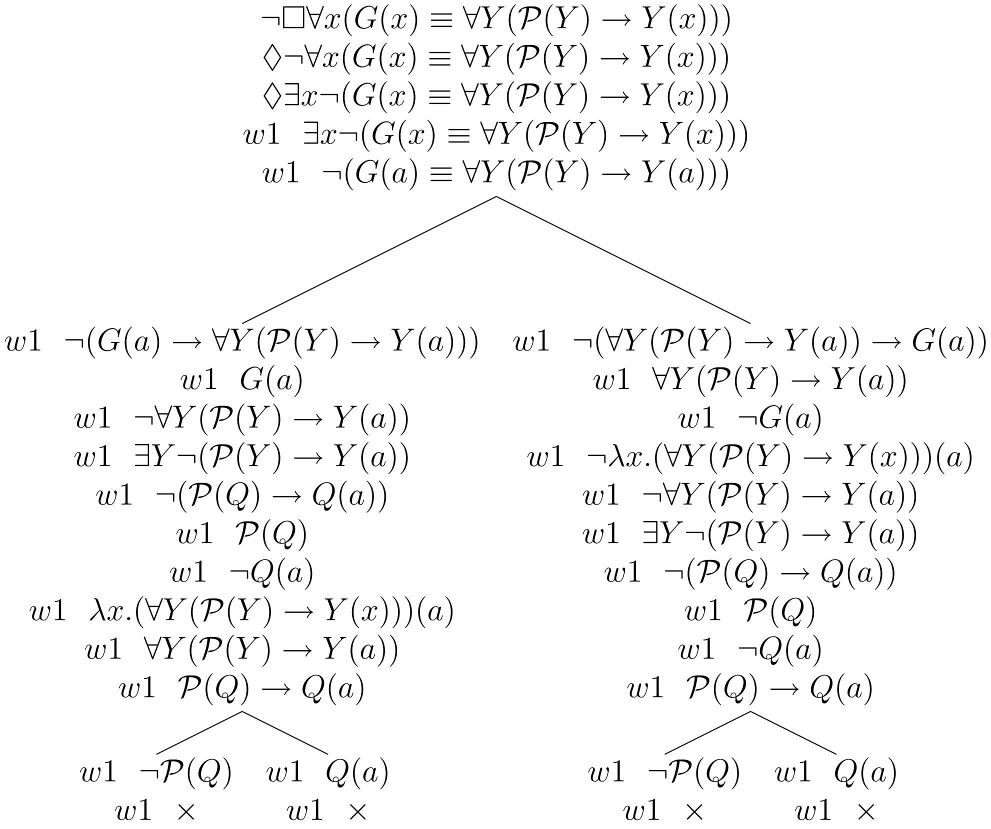
The derivability of $\mathcal{P}(G)$ from Axiom 3 would be intuitively clear because a property G is the collection of all positive properties. The formal proof is the following:



Given the definitions of *pos* and *intersection of*, both $\neg pos(\mathcal{P})$ and $\neg(G\ intersection\ of\ \mathcal{P})$ clearly lead to contradiction. However, we keep sticking to the formal proof here. Now $pos(\mathcal{P})$ being an abbreviation of $\neg\forall X(\mathcal{P}(X) \to \mathcal{P}(X))$, $\neg pos(\mathcal{P})$ turns out to be:

$$\begin{gathered}
\neg\forall X(\mathcal{P}(X) \to \mathcal{P}(X)) \\
\exists X\neg(\mathcal{P}(X) \to \mathcal{P}(X)) \\
\neg(\mathcal{P}(P) \to \mathcal{P}(P)) \\
\mathcal{P}(P) \\
\neg\mathcal{P}(P) \\
\times
\end{gathered}$$

In the same manner, $G\ intersection\ of\ \mathcal{P}$ and G being $\Box\forall x(G(x) \equiv \forall Y(\mathcal{P}(Y) \to Y(x)))$ and $\lambda x.(\forall Y(\mathcal{P}(Y) \to Y(x)))$ respectively, the tree is going to be:



All branches considered, the above tree is closed. Therefore, it is proved that the Axiom 3 implies $\mathcal{P}(G)$.

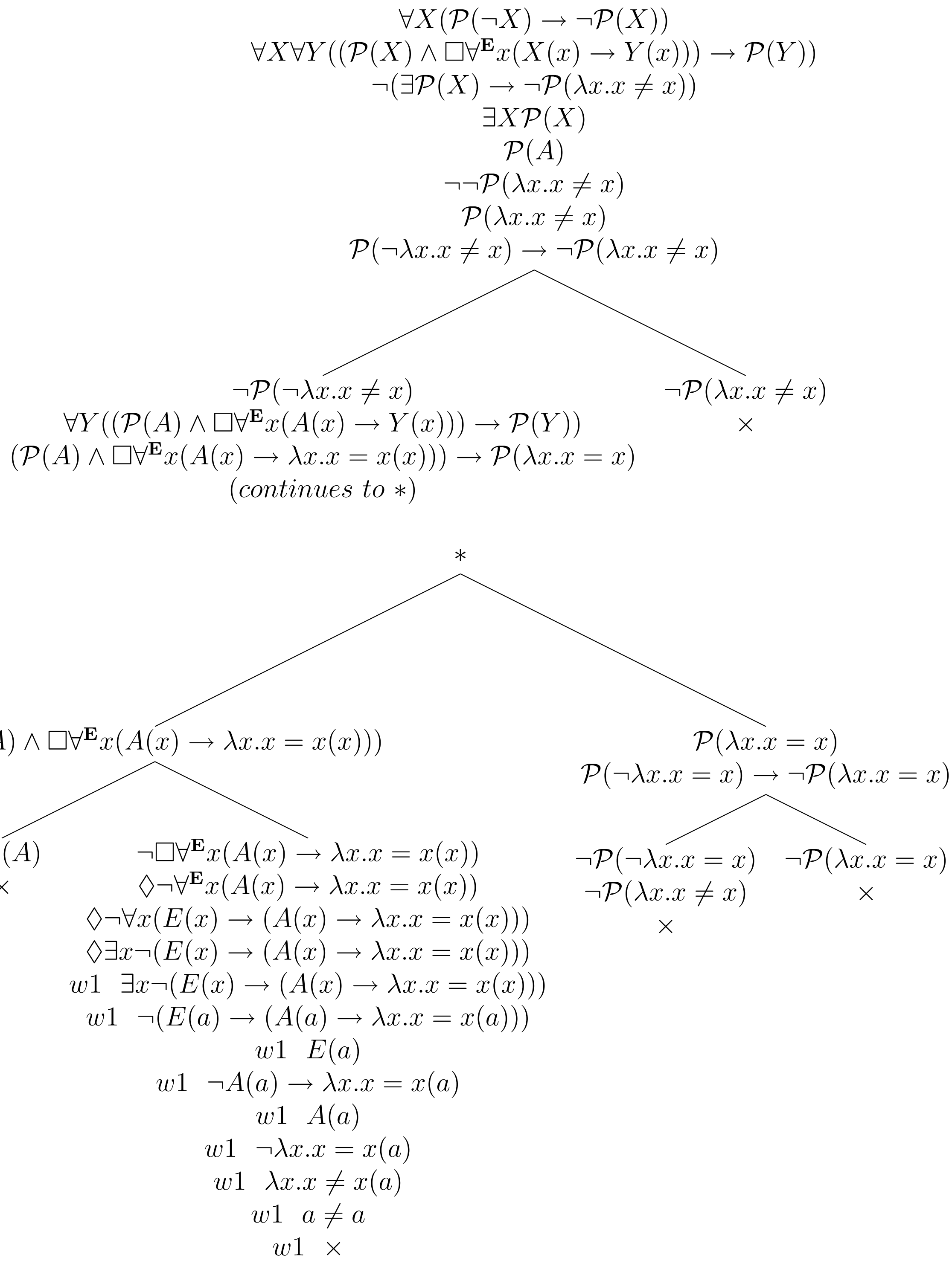
Next, we prove that $\mathcal{P}(G)$ implies $\Diamond\exists^{\mathbf{E}}xG(x)$. In order to prove it, we first prove formally the following proposition the informal version of which we have already seen in the previous section.

PROPOSITION 1 $\forall X(\mathcal{P}(X) \to \Diamond\exists^{\mathbf{E}}xX(x))$

Roughly speaking, the informal version of Proposition 1 asserts that any positive properties can be instantiated. More exactly, it means that if a property X is positive, an object which has this property can exist. The formal expression of this assertion is $\forall X(\mathcal{P}(X) \to \Diamond\exists^{\mathbf{E}}xX(x))$. We need the following lemma first to prove formally Proposition 1.

LEMMA 1 $\exists X\mathcal{P}(X) \to \neg\mathcal{P}(\lambda x.x \neq x)$

The validity of this lemma would be intuitively acceptable. If there exitst something positive, it must be identical to itself. The following tree is its proof (we assume Axiom 1 and Axiom 2).



With the above lemma and Axiom 2, we can prove Proposition 1 as follows.[4]

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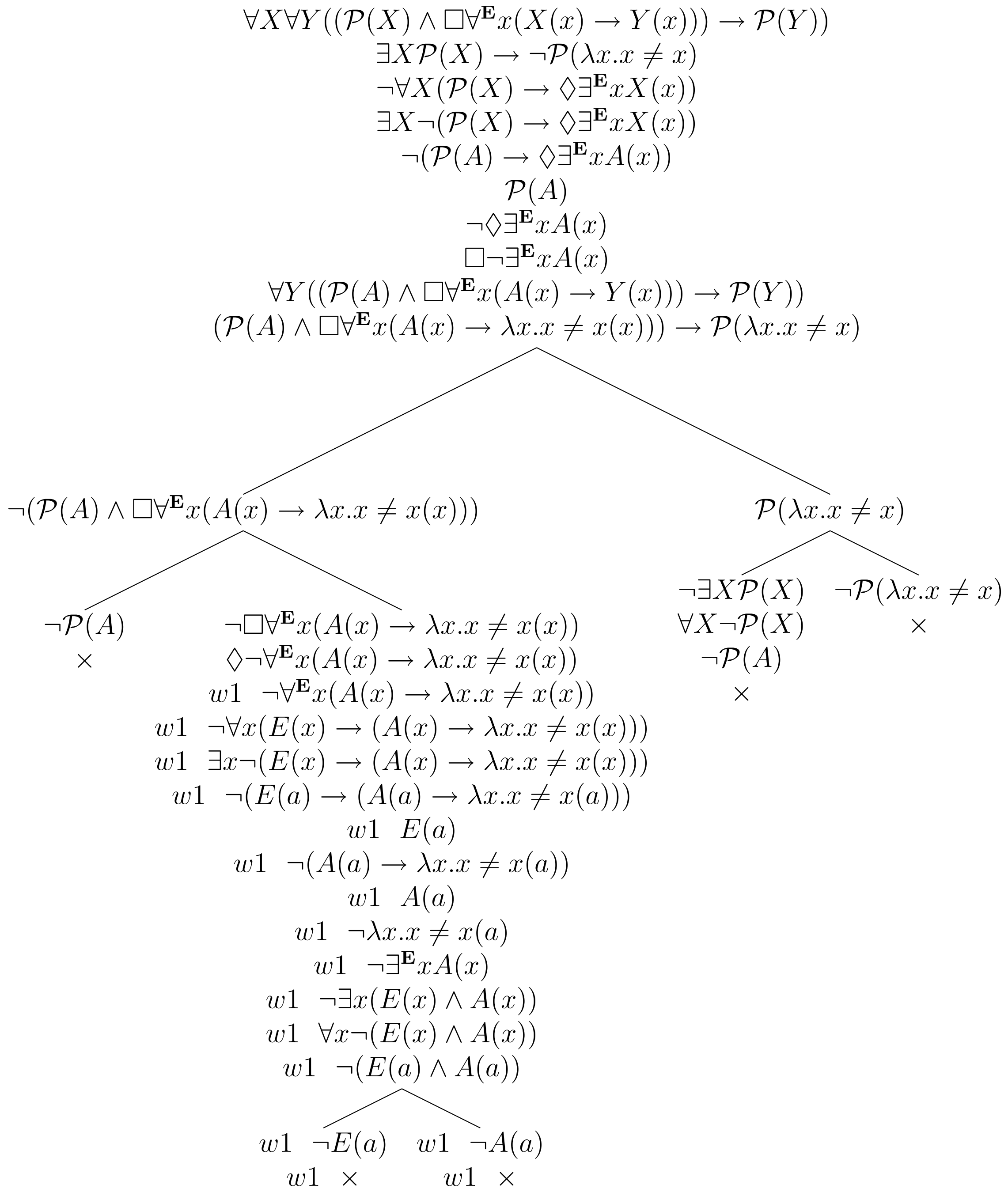
[4] In proving Proposition 1, I almost strictly follow Fitting because I could not find a more elegant proof than his proof (Fitting, p. 149).



$$\forall X \forall Y ((\mathcal{P}(X) \land \Box \forall^{\mathbf{E}} x (X(x) \to Y(x))) \to \mathcal{P}(Y))$$
$$\exists X \mathcal{P}(X) \to \neg \mathcal{P}(\lambda x. x \neq x)$$
$$\neg \forall X (\mathcal{P}(X) \to \Diamond \exists^{\mathbf{E}} x X(x))$$
$$\exists X \neg (\mathcal{P}(X) \to \Diamond \exists^{\mathbf{E}} x X(x))$$
$$\neg (\mathcal{P}(A) \to \Diamond \exists^{\mathbf{E}} x A(x))$$
$$\mathcal{P}(A)$$
$$\neg \Diamond \exists^{\mathbf{E}} x A(x)$$
$$\Box \neg \exists^{\mathbf{E}} x A(x)$$
$$\forall Y ((\mathcal{P}(A) \land \Box \forall^{\mathbf{E}} x (A(x) \to Y(x))) \to \mathcal{P}(Y))$$
$$(\mathcal{P}(A) \land \Box \forall^{\mathbf{E}} x (A(x) \to \lambda x. x \neq x(x))) \to \mathcal{P}(\lambda x. x \neq x)$$

$$\neg (\mathcal{P}(A) \land \Box \forall^{\mathbf{E}} x (A(x) \to \lambda x. x \neq x(x)))$$

$$\neg \mathcal{P}(A)$$
$$\times$$

$$\neg \Box \forall^{\mathbf{E}} x (A(x) \to \lambda x. x \neq x(x))$$
$$\Diamond \neg \forall^{\mathbf{E}} x (A(x) \to \lambda x. x \neq x(x))$$
$$w1 \quad \neg \forall^{\mathbf{E}} x (A(x) \to \lambda x. x \neq x(x))$$
$$w1 \quad \neg \forall x (E(x) \to (A(x) \to \lambda x. x \neq x(x)))$$
$$w1 \quad \exists x \neg (E(x) \to (A(x) \to \lambda x. x \neq x(x)))$$
$$w1 \quad \neg (E(a) \to (A(a) \to \lambda x. x \neq x(a)))$$
$$w1 \quad E(a)$$
$$w1 \quad \neg (A(a) \to \lambda x. x \neq x(a))$$
$$w1 \quad A(a)$$
$$w1 \quad \neg \lambda x. x \neq x(a)$$
$$w1 \quad \neg \exists^{\mathbf{E}} x A(x)$$
$$w1 \quad \neg \exists x (E(x) \land A(x))$$
$$w1 \quad \forall x \neg (E(x) \land A(x))$$
$$w1 \quad \neg (E(a) \land A(a))$$

$$w1 \quad \neg E(a)$$
$$w1 \quad \times$$

$$w1 \quad \neg A(a)$$
$$w1 \quad \times$$

$$\mathcal{P}(\lambda x. x \neq x)$$

$$\neg \exists X \mathcal{P}(X)$$
$$\forall X \neg \mathcal{P}(X)$$
$$\neg \mathcal{P}(A)$$
$$\times$$

$$\neg \mathcal{P}(\lambda x. x \neq x)$$
$$\times$$

Once we get the above results, we can rather easily prove that $\mathcal{P}(G)$ implies $\Diamond \exists^{\mathbf{E}} x G(x)$.

$$\mathcal{P}(G)$$
$$\forall X (\mathcal{P}(X) \to \Diamond \exists^{\mathbf{E}} x X(x))$$
$$\neg \Diamond \exists^{\mathbf{E}} x G(x)$$

$$\begin{array}{cc} \multicolumn{2}{c}{\mathcal{P}(G) \to \Diamond\exists^{\mathbf{E}} x G(x)} \\ \neg\mathcal{P}(G) & \Diamond\exists^{\mathbf{E}} x G(x) \\ \times & \times \end{array}$$

Finally, we prove that $\Diamond\exists^{\mathbf{E}} x G(x)$ implies Proposition 3. Seeing that $\exists^{\mathbf{E}} x G(x)$ is an abbreviation of $\exists(E(x) \wedge G(x))$, the implication is intuitively straightforward. The following is the formal proof.

$$\begin{array}{c} \Diamond\exists^{\mathbf{E}} x G(x) \\ \neg\Diamond\exists x G(x) \\ \Diamond\exists x (E(x) \wedge G(x)) \\ \Box\neg\exists x G(x) \\ w1 \;\; \exists x(E(x) \wedge G(x)) \\ w1 \;\; E(a) \wedge G(a) \\ w1 \;\; E(a) \\ w1 \;\; G(a) \\ w1 \;\; \neg\exists x G(x) \\ w1 \;\; \forall x \neg G(x) \\ w1 \;\; \neg G(a) \\ w1 \;\; \times \end{array}$$

All of the above taken into account, it has been formally shown that Axiom 3 (the recursiveness of any collection of positive properties) implies Proposition 3 (the possibility of the existence of God). To establish the equivalence between Axiom 3 and Proposition 3, next we move to the proof that Proposition 3 implies Axiom 3.

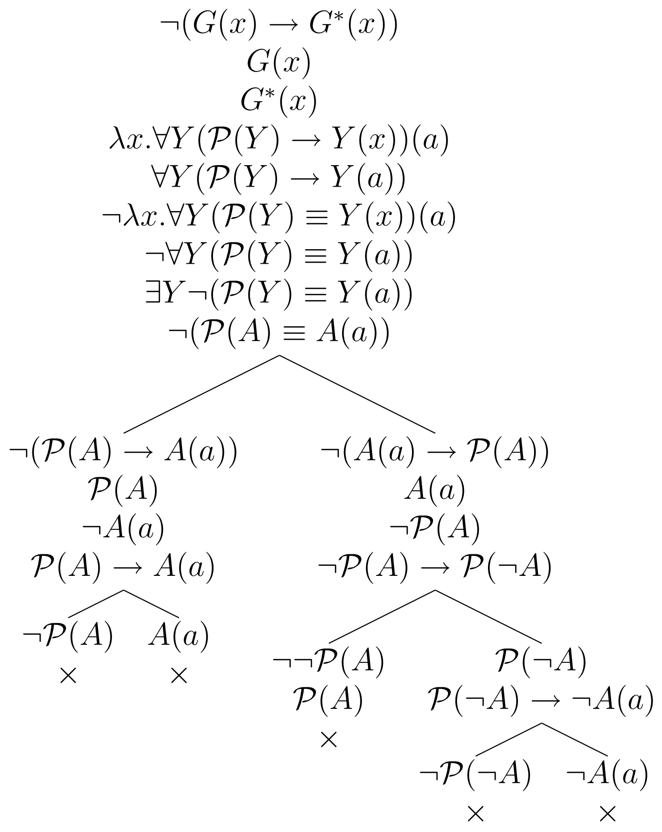
First, we directly prove that Proposition 3 implies $\mathcal{P}(G)$ by introducing another definition of being God. The following is the definition.

ANOTHER DEFINITION OF GODNESS G^* $\lambda x.\forall Y(\mathcal{P}(Y) \equiv Y(x))$

In addition to what the previous definition of Godness G says, this definition G^* also means that if something has some property, that property is positive. In fact, these two definition are equivalent. The following is its proof (we assume Axiom 1B).

$$\begin{array}{cc} \multicolumn{2}{c}{\forall X(\neg\mathcal{P}(X) \to \mathcal{P}(\neg X))} \\ \multicolumn{2}{c}{\neg\forall x(G(x) \equiv G^*(x))} \\ \multicolumn{2}{c}{\exists x \neg(G(x) \equiv G^*(x))} \\ \multicolumn{2}{c}{\neg(G(a) \equiv G^*(a))} \\ \neg(G(x) \to G^*(x)) & \neg(G^*(x) \to G(x)) \end{array}$$

First, we construct the left side of the tree.



The next is the right side.

$$\begin{gathered}
\neg(G^*(x) \to G(x)) \\
\lambda x.\forall Y(\mathcal{P}(Y) \equiv Y(x))(a) \\
\forall Y(\mathcal{P}(Y) \equiv Y(a)) \\
\neg\lambda x.\forall Y(\mathcal{P}(Y) \to Y(x))(a) \\
\neg\forall Y(\mathcal{P}(Y) \to Y(a)) \\
\exists Y\neg(\mathcal{P}(Y) \to Y(a)) \\
\neg(\mathcal{P}(A) \to A(a)) \\
\mathcal{P}(a) \\
\neg A(a) \\
\mathcal{P}(A) \equiv A(a) \\
\mathcal{P}(A) \to A(a) \\
A(a) \to \mathcal{P}(A) \\
\times
\end{gathered}$$

In addition to this equivalence $\forall x(G(x) \equiv G^*(x))$, we need the formal version of Axiom 4 ("Any positive property is necessarily so, and any negative property is necessarily so").

AXIOM 4

A $\forall X(\mathcal{P}(X) \to \Box\mathcal{P}(X))$

B $\forall X(\neg\mathcal{P}(X) \to \Box\neg\mathcal{P}(X))$[5]

With the necessitated version of $\forall x(G(x) \equiv G^*(x))$[6] and Axiom 4, we can prove that $\Diamond\exists G(x)$ implies $\mathcal{P}(G)$.

$$\begin{array}{c}
\Diamond\exists x G(x)\\
\Box\forall x(G(x) \equiv G^*(x))\\
\forall X(\neg\mathcal{P}(X) \to \Box\neg\mathcal{P}(X))\\
\neg\mathcal{P}(G)\\
\neg\mathcal{P}(G) \to \Box\neg\mathcal{P}(G)
\end{array}$$

$$\begin{array}{cc}
\neg\neg\mathcal{P}(G) & \Box\neg\mathcal{P}(G)\\
\mathcal{P}(G) & w1\ \ \exists x G(x)\\
\times & w1\ \ G(a)\\
 & w1\ \ \forall x(G(x) \equiv G^*(x))\\
 & w1\ \ G(a) \equiv G^*(a)\\
 & w1\ \ G^*(a)\\
 & w1\ \ \lambda x.\forall Y(\mathcal{P}(Y) \to Y(x))(a)\\
 & w1\ \ \forall Y(\mathcal{P}(Y) \to Y(a))\\
 & w1\ \ \mathcal{P}(G) \equiv G(a)\\
 & w1\ \ \mathcal{P}(G)\\
 & w1\ \ \neg\mathcal{P}(G)\\
 & w1\ \ \times
\end{array}$$

Now the only remaining task to prove the equivalence between Axiom 3 and Proposition 3 is to prove that $\mathcal{P}(G)$ implies Proposition 3. The proof is a bit lengthy. In the following proof, we assume Axiom 2.

$$\begin{array}{c}
\mathcal{P}(G)\\
\forall X\forall Y((\mathcal{P}(X) \wedge \Box\forall^{\mathbf{E}}x(X(x) \to Y(x))) \to \mathcal{P}(Y))\\
\neg\forall\mathcal{Z}(pos(\mathcal{Z}) \to \forall X((X\ \mathit{intersection\ of}\ \mathcal{Z}) \to \mathcal{P}(X)))\\
\exists\mathcal{Z}\neg(pos(\mathcal{Z}) \to \forall X((X\ \mathit{intersection\ of}\ \mathcal{Z}) \to \mathcal{P}(X)))\\
\neg(pos(A) \to \forall X((X\ \mathit{intersection\ of}\ A) \to \mathcal{P}(X)))\\
pos(A)\\
\forall X(A(X) \to \mathcal{P}(X))\\
\neg\forall X((X\ \mathit{intersection\ of}\ A) \to \mathcal{P}(X))\\
\exists X\neg((X\ \mathit{intersection\ of}\ A) \to \mathcal{P}(X))\\
\neg((B\ \mathit{intersection\ of}\ A) \to \mathcal{P}(B))
\end{array}$$

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[5] In fact, Axiom 4B can be deducible from Axiom 4A by assuming Axiom 1. However, we omit its proof here.

[6] An intuitive justification of $\Box\forall x(G(x) \equiv G^*(x))$ is that God has to be God in all possible worlds. Thought we can in fact deduce the necessitated version of $\forall x(G(x) \equiv G^*(x))$, we omit the deduction.

B intersection of A